Riemann-Hilbert problem for the small dispersion limit of the KdV equation and linear overdetermined systems of Euler-Poisson-Darboux type

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Abstract

We study the Cauchy problem for the Korteweg de Vries (KdV) equation with small dispersion and with monotonically increasing initial data using the Riemann-Hilbert (RH) approach. The solution of the Cauchy problem, in the zero dispersion limit, is obtained using the steepest descent method for oscillatory Riemann-Hilbert problems. The asymptotic solution is completely described by a scalar function $\mathcal{G}$ that satisfies a scalar RH problem and a set of algebraic equations constrained by algebraic inequalities. The scalar function $\mathcal{G}$ is equivalent to the solution of the Lax-Levermore maximization problem. The solution of the set of algebraic equations satisfies the Whitham equations. We show that the scalar function $\mathcal{G}$ and the Lax-Levermore maximizer can be expressed as the solution of a linear overdetermined system of equations of Euler-Poisson-Darboux type. We also show that the set of algebraic equations and algebraic inequalities can be expressed in terms of solutions of a different set of linear overdetermined systems of equations of Euler-Poisson-Darboux type. Furthermore we show that the set of algebraic equations is equivalent to the classical solution of the Whitham equations expressed by the hodograph transformation.

1 Introduction

The Cauchy problem for the Korteweg de Vries (KdV) equation

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \quad u(x,0) = u_0(x), \quad \epsilon > 0 \tag{1.1}$$

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in the zero-dispersion limit has been widely studied. The physical interest in this limit is due to the fact that it describes the phenomenon of shock waves in dissipationsless dispersive media. Dispersive shock waves are characterized by the appearance of rapid modulated oscillations. Gurevich and Pitevskii [1] suggested that these oscillations could be modeled by the solution of the one-phase Whitham equations [2]. The multiphase or g-phase Whitham equations were derived by Flaschka, Forest and Mc Laughlin [3]. Lax and Levermore [4] rigorously showed that the multiphase Whitham equations appear in the zero dispersion limit of the Cauchy problem for the KdV equation with asymptotically reflectionless initial data. Later Venakides [5] considered a wider class of initial data. Lax and Levermore developed their theory in the frame of the zero-dispersion asymptotics for the solution of the inverse scattering problem of KdV. They showed that the principal term of the relevant asymptotics is given by the g-phase solution [6] of the KdV equation with the wave parameters depending on the functions $u_1(x,t) > \cdots > u_{2g+1}(x,t)$ which satisfy the g-phase Whitham equations:

$$\frac{\partial u_i}{\partial t} - v_i(u_1, u_2, \ldots, u_{2g+1})\frac{\partial u_i}{\partial x} = 0, \quad x, t, u_i \in \mathbb{R}, \quad i = 1, ..., 2g+1, \ g \geq 0. \tag{1.2}$$

For $g > 0$ the speeds $v_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, 2, \ldots, 2g+1$, depend through $u_1, \ldots, u_{2g+1}$ on complete hyperelliptic integrals of genus g. For $g = 0$ we define $u_1 = u$ and the zero-phase Whitham equation reads

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} = 0 \tag{1.3}$$

and can be integrated by the method of characteristics. The formal integrability of equations (1.2) for $g > 0$ was obtained by Tsarev [7] using the geometric-Hamiltonian structure [8] of the Whitham equations. Namely he proved that if the functions $w_i = w_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$, solve the linear over-determined system of equations

$$\frac{\partial w_i}{\partial u_j} = \frac{1}{v_i - v_j}\frac{\partial v_i}{\partial u_j}[w_i - w_j], \quad i, j = 1, 2, \ldots, 2g+1, \ \ i \neq j, \tag{1.4}$$

where $v_i = v_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$, are the speeds in (1.2), then the solution $\vec{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_{2g+1}(x,t))$ of the so called hodograph transformation

$$x = -v_i(\vec{u})\, t + w_i(\vec{u}) \quad i = 1, \ldots, 2g+1\,, \tag{1.5}$$

satisfies equations (1.2). Conversely, any solution $(u_1(x,t), u_2(x,t), \ldots, u_{2g+1}(x,t))$ of (1.2) can be obtained in this way in the neighborhood of (x_0, t_0) at which u_{ix}'s are not vanishing. The general solution of the Tsarev equations was obtained in [9],[10],[11] for monotonically increasing initial data. The key step introduced in [9],[11] was to reduce the solution of the Tsarev system to the solution of linear-overdetermined systems of Euler-Poisson Darboux type for some functions $q_k = q_k(\vec{u})$, $k = 1, \ldots, g$, namely

$$\frac{\partial}{\partial u_i}q_k - \frac{\partial}{\partial u_j}q_k = 2(u_i - u_j)\frac{\partial^2}{\partial u_i \partial u_j}q_k, \quad i \neq i, \ i, j = 1, \ldots, 2g+1.$$

In this paper we show that these functions play a role in the Lax-Levermore maximization problem. We study the small dispersion limit of KdV for monotonically increasing analytic initial data bounded at infinity.

We use a different approach to the small dispersion limit obtained in [12] by Deift, Venakides and Zhou. These authors first used the formulation of the Cauchy problem for KdV as a Riemann-Hilbert (RH) problem [18] (see also [14]). Then, for computing the small ϵ-asymptotics, they used the steepest descent method for oscillatory Riemann-Hilbert problems introduced in [16]. Their procedure leads to a scalar RH problem for a certain phase function $\mathcal{G}$ that turns out to be equivalent to the solution of the leading order variational problem in the Lax-Levermore theory. Furthermore they reduce the initial value problem for the Whitham equations to solving a set of algebraic equations constrained by algebraic inequalities. Existence and uniqueness of the initial value problem for the Whitham equations follows from the existence and uniqueness of the solution of the variational problem in the Lax-Levermore theory.

The RH problem for the scalar function $\mathcal{G}$ is well defined even for smooth initial data.

We show that, for smooth monotonically increasing initial data bounded at infinity, the function $\mathcal{G}$ and the Lax-Levermore maximizer can be expressed as the solution of a linear overdetermined system of Euler-Poisson-Darboux type. In the same way, the set of algebraic equations obtained through the Deift, Venakides and Zhou approach can be expressed as the solution of a set of linear overdetermined systems of equations of Euler-Poisson Darboux type. We also show that this set of algebraic equations is equivalent to the set of algebraic equations defined by the hodograph transformation (1.5).

The advantage of this representation is clear when one tries to construct effectively the solution of the set of algebraic equations constrained by the algebraic inequalities. It is simpler to evaluate space derivatives and to estimate the sign of the quantities in the inequalities. Indeed this representation was used in [10],[17] to give an upper bound to the genus of the solution of the Whitham equations.

This paper is organized as follows.

Section 2 contains the definitions of the Abelian differentials on Riemann surface and the meromorphic analogue of the Cauchy kernel.

In Section 3 we review the Riemann-Hilbert steepest descent method for the zero dispersion KdV equation with monotonically increasing initial data.

We determine the scalar function $\mathcal{G}$ associated to the RH problem and all its properties in section 4. We show that the set of algebraic equations that we call moment conditions and normalization conditions are equivalent to the set of algebraic equations defined by the hodograph transformation.

In Section 5 we show that the hodograph transformation obtained in Sec 4 is equivalent to the classical one provided in [9] and [10]. We then show that the moment conditions, the normalization conditions and the Lax-Levermore maximizer can be expressed in terms of solutions of linear overdetermined systems of Euler-Poisson-Darboux type. We also derive in a simple way the equations which determine the phase transitions.

Finally in section 6 we summarize the main results and draw our conclusions.

2 Riemann surfaces and Abelian differentials: notations and definitions

Let

$$\mathcal{S}_g := \left\{ P = (\lambda, y),\ y^2 = \prod_{j=1}^{2g+1} (\lambda - u_j) \right\}, \quad u_1 > u_2 > \cdots > u_{2g+1}\,, \quad u_i \in \mathbb{R}\,, \tag{2.1}$$

be the hyperelliptic Riemann surface of genus $g \geq 0$. We shall use the standard representation of $\mathcal{S}_g$ as a two-sheeted covering of $C\mathbb{P}^1$ with cuts along the intervals

$$[u_{2k}, u_{2k-1}], \quad k = 1, \dots, g+1, \quad u_{2g+2} = -\infty\,. \tag{2.2}$$

We choose the basis $\{\alpha_j, \beta_j\}_{j=1}^g$ of the homology group $H_1(\mathcal{S}_g)$ so that α_j lies fully on the upper sheet and encircles clockwise the interval $[u_{2j}, u_{2j-1}]$, $j = 1, \dots, g$, while β_j emerges on the upper sheet on the cut $[u_{2j}, u_{2j-1}]$, passes anti-clockwise to the lower sheet trough the cut $(-\infty, u_{2g+1}]$ and returns to the initial point through the lower sheet.

The one-forms that are analytic on the closed Riemann surface $\mathcal{S}_g$ except for a finite number of points are called Abelian differentials.

We define on $\mathcal{S}_g$ the following differentials [18]:

1) The canonical basis of holomorphic one-forms or Abelian differentials of the first kind $\phi_1, \phi_2 \dots \phi_g$:

$$\phi_k(\lambda) = \frac{\lambda^{g-1}\gamma_1^k + \lambda^{g-2}\gamma_2^k + \cdots + \gamma_g^k}{y(\lambda)} d\lambda\,, \quad k = 1, \dots, g\,. \tag{2.3}$$

The constants γ_i^k are uniquely determined by the normalization conditions

$$\int_{\alpha_j} \phi_k = \delta_{jk}\,, \quad j, k = 1, \dots, g. \tag{2.4}$$

We remark that an holomorphic differential having all its α-periods equal to zero is identically zero [18].

2) The set σ_k^g, $k \geq 0$, $g \geq 0$, of Abelian differentials of the second kind with a pole of order $2k+2$ at infinity, with asymptotic behavior

$$\sigma_k^g(\lambda) = \left[\lambda^{k-\frac{1}{2}} + O(\lambda^{-\frac{3}{2}})\right] d\lambda \quad \text{for large } \lambda \tag{2.5}$$

and normalized by the condition

$$\int_{\alpha_j} \sigma_k^g = 0, \quad j = 1, \dots, g\,. \tag{2.6}$$

We use the notation

$$\sigma_0^g(\lambda) = dp^g(\lambda)\,, \qquad 12\sigma_1(\lambda) = dq^g(\lambda) \quad g \geq 0. \tag{2.7}$$

In literature the differentials $dp^g(\lambda)$ and $dq^g(\lambda)$ are called quasi-momentum and quasi-energy respectively [8]. The explicit formula for the differentials σ_k^g, $k \geq 0$, is given by the expression

$$\sigma_k^g(\lambda) = \frac{P_k^g(\lambda)}{y(\lambda)} d\lambda\,, \quad P_k^g(\lambda) = \lambda^{g+k} + a_1^k \lambda^{g+k-1} + a_2^k \lambda^{g+k-2} \cdots + a_{g+k}^k\,, \tag{2.8}$$

where the coefficients $a_i^k = a_i^k(\vec{u})$, $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, g+k$, are uniquely determined by (2.5) and (2.6).

3) The Abelian differential of the third kind $\omega_{qq_0}(\lambda)$ with first order poles at the points $Q = (q, y(q))$ and $Q_0 = (q_0, y(q_0))$ with residues ± 1 respectively. Its periods are normalized by the relation

$$\int_{\alpha_j} \omega_{qq_0}(\lambda) = 0, \quad j = 1, \ldots, g\,. \tag{2.9}$$

In the following we mainly use the normalized differential $\omega_z^g(\lambda)$ which has simple poles at the points $Q^{\pm}(z) = (z, \pm y(z))$ with residue ± 1 respectively.

The differential $\omega_z^g(\lambda)$ is explicitly given by the expression

$$\omega_z^g(\lambda) = \frac{d\lambda}{y(\lambda)} \frac{y(z)}{\lambda - z} - \sum_{k=1}^{g} \phi_k(\lambda) \int_{\alpha_k} \frac{dt}{y(t)} \frac{y(z)}{t - z}\,, \tag{2.10}$$

where $\phi_k(\lambda)$, $k = 1, \ldots, g$, is the normalized basis of holomorphic differentials. By construction

$$\int_{\alpha_j} \omega_z^g(\lambda) = 0, \quad j = 1, \ldots, g. \tag{2.11}$$

The differential $\omega_z^g(\lambda)$ can also be written in the form

$$\omega_z^g(\lambda) = \frac{d\lambda}{y(\lambda)} \frac{y(z)}{\lambda - z} - \sum_{k=1}^{g} N_k^g(z, \vec{u}) \frac{\lambda^{g-k}}{y(\lambda)} d\lambda\,, \tag{2.12}$$

where

$$N_k^g(z, \vec{u}) = y(z) \sum_{j=1}^{g} \gamma_k^j \int_{\alpha_j} \frac{d\eta}{y(\eta)(\eta - z)}. \tag{2.13}$$

$\omega_z^g(\lambda)$ as a function of z, is an Abelian integral. The periods of this integral are obtained from the relations [19]

$$\int_{\alpha_j} d_z[\omega_z^g(\lambda)] = 0, \quad \int_{\beta_j} d_z[\omega_z^g(\lambda)] = 4\pi i \phi_k(\lambda)\,, \quad j = 1, \ldots g\,. \tag{2.14}$$

The differential $\omega_z^g(\lambda)$ satisfies the property [19]

$$d_z \omega_z^g(\lambda) = d_\lambda \omega_\lambda(z). \tag{2.15}$$

In the following we will use the single value restriction of $\omega_z^g(\lambda)$ determined by the conditions

$$\omega_z^g(\lambda)|_{z=u_{2g+1}} = 0$$

and by choosing $y(z)$ to be analytic off the cuts (2.2) and real positive $z > u_1$. We will still denote this single value restriction with $\omega_z^g(\lambda)$. We remark that $\omega_z^g(\lambda)$ is a meromorphic analogue of the Cauchy kernel on the Riemann surface $\mathcal{S}_g$ [19].

The next proposition is also important for our subsequent considerations.

Proposition 2.1 *[17] The Abelian differentials of the second kind $\sigma_k^g(\lambda)$, $k \geq 0$, defined in (2.5) satisfy the relations*

$$\sigma_k^g(\lambda) = \frac{1}{2}\operatorname*{Res}_{z=\infty}\left[\omega_z^g(\lambda)\, z^{k-\frac{1}{2}}dz\right] = -\frac{1}{2k+1}d_\lambda \operatorname*{Res}_{z=\infty}\left[\omega_\lambda^g(z)\, z^{k+\frac{1}{2}}\right], \tag{2.16}$$

where $\omega_z^g(\lambda)$ has been defined in (2.10), $\omega_\lambda^g(z)$ is the normalized Abelian differential of the third kind with simple poles at the points $Q^\pm(\lambda) = (\lambda, \pm y(\lambda))$ with residue ± 1 respectively and d_λ denotes differentiation with respect to λ.

3 Riemann-Hilbert steepest descent method for the zero dispersion KdV equation with monotonically increasing initial data

Following [18] we reformulate the inverse scattering for the KdV equation as a RH problem. We consider monotonically increasing analytic initial data $u_0(x)$ bounded at infinity. For convenience we assume

$$\lim_{x\to-\infty} u_0(x) = 0, \quad \lim_{x\to+\infty} u_0(x) = 1.$$

We suppose

$$\int_{-\infty}^{c} u_0(x)(1+|x|^{2+\delta})dx < \infty, \quad \int_{c}^{-\infty}(1-u_0(x))(1+|x|^{2+\delta})dx < \infty, \tag{3.1}$$

for all finite c and $\delta > 0$.
Let $r(\lambda;\epsilon)$, $\lambda > 0$, be the reflection coefficient from the left of the Schödinger equation $-\epsilon^2 f_{xx} + u_0(x)f = \lambda f$. Define the matrix [12], [15]

$$\nu(\lambda,\epsilon) = \begin{cases} \sigma_1, & \lambda < 0 \\ \begin{pmatrix} 0 & -\bar{r}e^{-2i\alpha/\epsilon} \\ re^{2i\alpha/\epsilon} & 1 \end{pmatrix}, & 0 < \lambda < 1, \\ \begin{pmatrix} 1-|r|^2 & -\bar{r}e^{-2i\alpha/\epsilon} \\ re^{2i\alpha/\epsilon} & 1 \end{pmatrix}, & \lambda > 1, \end{cases}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\alpha = 4t\lambda^{\frac{3}{2}} + x\lambda^{\frac{1}{2}}$. The goal is to find a row vector valued function $m(\lambda) = m(\lambda; x, t, \epsilon) = (m_1, m_2)$ analytic for complex λ off the real axis, satisfying the jump and asymptotic conditions

$$\begin{aligned} m_+(\lambda; x, t, \epsilon) =& m_-(\lambda; x, t, \epsilon)\nu(\lambda; x, t, \epsilon) \\ m(\lambda; x, t, \epsilon) \to& (1, 1) \quad \text{as } \lambda \to \infty, \end{aligned}$$

where $m_\pm = \lim_{\delta\to 0} m(\lambda \pm i\delta; x, t, \epsilon)$. The RH problem of finding the matrix $m(\lambda)$ given $\nu(\lambda)$ has a unique solution in the space $(1, 1) + L^2(d\lambda^{\frac{1}{2}})$. The solution of the Cauchy problem (1.1) is given by

$$u(x, t, \epsilon) = -2i\epsilon\partial_x m_{11}(x, t, \epsilon) \tag{3.2}$$

where

$$m_1(\lambda; x, t, \epsilon) = 1 + m_{11}\lambda^{-\frac{1}{2}} + O(\lambda^{-1}), \quad \lambda \to \infty, \tag{3.3}$$

see again ([18], [14]). In this paper we study the Cauchy problem (1.1) in the limit $\epsilon \to 0$, the so called zero-dispersion limit of the KdV equation. We use the WKB approximation with one turning point to calculate the reflection coefficient [20]

$$\begin{aligned} r(\lambda; \epsilon) &\simeq -ie^{-2i\rho(\lambda)/\epsilon}\chi_{[0,1]}(\lambda) \\ \rho(\lambda) &\simeq \lambda^{\frac{1}{2}}x(\lambda) - \int_{-\infty}^{x(\lambda)} [\lambda^{\frac{1}{2}} - (\lambda - u_0(x))^{\frac{1}{2}}]dx, \end{aligned}$$

where the quantity $x(\lambda)$ is defined by the relation $u_0(x(\lambda)) = \lambda$. As usual $\chi_{[0,1]}(\lambda)$ denotes the characteristic function of the interval $[0, 1]$. From the above considerations the jump matrix reduces to the identity matrix for $\lambda > 1$ and our RH problem is reduced to the interval $(-\infty, 1]$. The quantity $\rho(\lambda)$ can be expressed also in the form

$$\rho(\lambda) = \frac{1}{2}\int_0^\lambda \frac{f(y)}{\sqrt{\lambda - y}}dy \tag{3.4}$$

where $f(u)|_{t=0}$ is the inverse function of the initial data $u_0(x)$. In the following we identify r with its WKB approximation.

Following the procedure in [12], we introduce a change of the dependent variable m

$$M(\lambda) = m(\lambda)e^{\mathcal{G}(\lambda)\sigma_3/\epsilon} \tag{3.5}$$

where the scalar function $\mathcal{G}(\lambda) = \mathcal{G}(\lambda; x, t)$ is analytic in λ off the line $(-\infty, 1]$ and satisfies $\mathcal{G}_+(\lambda) + \mathcal{G}_-(\lambda) = 0$, for $\lambda \in (-\infty, 0)$ and $\mathcal{G}(\lambda) \to\ 0$ as $\lambda \to \infty$. From (3.5) and (3.2) we obtain

$$u(x, t) = -2i\epsilon\partial_x M_{11}(x, t) - 2\partial_x \mathcal{G}_1(x, t),$$

where $\mathcal{G}(\lambda) = \mathcal{G}_1(x,t)/\sqrt{\lambda} + O(1/\lambda)$ and $M_1(\lambda; x, t) = 1 + M_{11}(x,t)/\lambda^{\frac{1}{2}}$. The RH problem in the new variable becomes $M_+(\lambda) = M_-(\lambda)\Upsilon(\lambda)$, $\lambda \in (0,1)$, where

$$\Upsilon = \begin{pmatrix} 0 & -ie^{-ih/\epsilon} \\ -ie^{ih/\epsilon} & e^{-i(\mathcal{G}_+ - \mathcal{G}_-)/\epsilon} \end{pmatrix}$$

and $h = \mathcal{G}_+ + \mathcal{G}_- - 2\rho + 2\alpha$. For computing the small ϵ asymptotics we follow the technique in [12]. The interval $0 < \lambda < 1$ is partitioned into finitely many intervals $I_j = (u_{2j}, u_{2j-1})$, $j = 1, \ldots, g+1$ and $0 = u_{2g+2} < u_{2g+1} < \cdots < u_2 < u_1 < 1$, $g \geq 0$. Using the steepest descent method [12],[16], the jump matrix Υ can be reduced to one of the two forms with exponentially small errors as $\epsilon \searrow 0$,

$$\begin{aligned} &a0) \quad \begin{pmatrix} 0 & -ie^{-ih/\epsilon} \\ -ie^{ih/\epsilon} & 0 \end{pmatrix} \quad \lambda \in \cup_{j=1}^{g+1} I_j, \\ &b0) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in (0,1)\backslash \cup_{j=1}^{g+1} I_j. \end{aligned} \tag{3.6}$$

The function $\mathcal{G}$ satisfies the following conditions:

$$\frac{(\mathcal{G}_+ - \mathcal{G}_-)}{2i} < 0, \quad \text{and } h'(\lambda) = 0, \text{ thus } \mathcal{G}_+ + \mathcal{G}_- - 2\rho + 2\alpha = -\Omega_j, \quad \lambda \in \cup_{j=1}^{g+1} I_j, \tag{3.7}$$

where Ω_j is some constant of integration;

$$\mathcal{G}_+ - \mathcal{G}_- = 0 \quad \text{and} \quad h'(\lambda) > 0, \quad \lambda \in (0,1)\backslash \cup_{j=1}^{g+1} (u_{2j}, u_{2j-1}). \tag{3.8}$$

The remaining RH problem is the following

$$\begin{aligned} M_+ &= M_-\sigma_1, \quad \lambda \in (-\infty, 0), \\ M_+ &= -iM_-\sigma_1 e^{-i\sigma_3\Omega_j/\epsilon}, \quad \lambda \in \cup_{j=1}^{g+1} I_j, \end{aligned} \tag{3.9}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

To complete the solution of the initial value problem (1.1) we have to determine the phase function $\mathcal{G}$, the intervals I_j, $j = 1, \ldots, g+1$, the values Ω_j. Once all these steps have been made, we have to solve the RH problem (3.9), so that the solution of the initial value problem (1.1) can be expressed by [12], [21]

$$u(x,t,\epsilon) = \sum_{j=1}^{2g+1} u_j + 2a_1^0 - 2\epsilon^2 \frac{\partial^2}{\partial x^2} \log \theta(\vec{\Omega}/(2\pi\epsilon)),$$

where a_1^0 has been defined in (2.8) and $\vec{\Omega} = (\Omega_1, \Omega_2, \ldots, \Omega_g)$. The theta function is defined by

$$\theta(z, B) = \sum_{m \in Z^g} e^{2\pi i(m,z) + \pi i(m,Bm)}, \quad z \in C^g,$$

where B is the period matrix of the holomorphic differentials (2.3), namely $B_{ij} = \int_{\beta_j} \phi_i$, $i, j = 1, \ldots, g$.

4 Determination of $\mathcal{G}(\lambda)$

We observe that for each fix x and t the function $\mathcal{G}'(\lambda)$ satisfies the following RH problem

$$\begin{aligned}
\mathcal{G}'_+ + \mathcal{G}'_- &= 0, \quad \lambda \in (-\infty, 0) \\
\mathcal{G}'_+ - \mathcal{G}'_- &= 0, \quad \lambda > 1 \text{ and } \lambda \in (0,1) - \mathcal{L}_g, \\
\mathcal{G}'_+ + \mathcal{G}'_- - 2\rho' + 2\alpha' &= 0, \quad \lambda \in \mathcal{L}_g,
\end{aligned} \tag{4.10}$$

where $\mathcal{L}_g = \cup_{j=1}^{g+1} I_j$. We call the intervals I_j bands, while the intervals (u_{2j+1}, u_{2j}), $j = 0, \ldots, g$, $u_0 = 1$, are called gaps. We add the requirement that

$$(\sqrt{\lambda}\mathcal{G}'(\lambda))_\pm, \text{ are continuous functions for real } \lambda. \tag{4.11}$$

It follows that $\mathcal{G}'(\lambda)_\pm$ are continuous functions for $\lambda \in (0,1)$.

We recall that the condition $\mathcal{G}(\lambda) = O(\lambda^{-\frac{1}{2}})$ for large λ, implies

$$\mathcal{G}'(\lambda) = O(\lambda^{-\frac{3}{2}}). \tag{4.12}$$

We observe that the condition (3.8) implies the following normalization condition

$$\int_{\alpha_j} \mathcal{G}'(\lambda) d\lambda = 0, \quad j = 1, \ldots, g, \tag{4.13}$$

where α_j is any clockwise close loop around the cut I_j. When the loops α_j collapse to the interval I_j described twice, the conditions (4.13) become

$$\int_{I_j} (\mathcal{G}'_+ - \mathcal{G}'_-) d\lambda = 0, \quad j = 1, \ldots, g. \tag{4.14}$$

The solution of the RH problem (4.10) that satisfies (4.11) is given by the integral [22]

$$\mathcal{G}'(\lambda) = \frac{y(\lambda)}{2\pi i} \int_{\mathcal{L}_g} \frac{(2\rho'(z) - 2\alpha'(z))dz}{(z - \lambda)y^+(z)} \tag{4.15}$$

where

$$y^2 = \prod_{j=1}^{2g+1} (\lambda - u_j)\,. \tag{4.16}$$

We choose $y(\lambda)$ to be analytic off the intervals $(-\infty,0]\cup\mathcal{L}_g$, real and positive for $\lambda>u_1$ and we denote $y^+(\lambda)$ the boundary value from above the cut $(-\infty,0]\cup\mathcal{L}_g$.

In order for (4.15) to satisfy (4.12) we must impose the following moment conditions

$$\int_{\mathcal{L}_g}\frac{\rho'(\lambda)-\alpha'(\lambda))}{y^+(\lambda)}\lambda^k d\lambda=0,\quad k=0,\dots,g. \tag{4.17}$$

Furthermore we must impose the normalization conditions (4.13). We observe that (4.13) and (4.17) represent a system of $2g+1$ algebraic equations which, in principle, determines the end-points $u_1,u_1,\dots,u_{2g+1}$ of the intervals I_j, $j=1,\dots,g+1$.
For $g=0$ the moment condition reduces to the form

$$\frac{2}{\pi i}\int_0^u\frac{\rho'(\lambda)-\alpha'(\lambda))}{\sqrt{\lambda-u}}d\lambda=f(u)-6tu-x=0,$$

which is the solution of the zero-phase equation (1.3).
We observe that

$$\mathcal{G}'_+(z_0)-\mathcal{G}'_-(z_0)=\mathcal{G}(z_0)=\frac{y(z_0)}{\pi i}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))dz}{(z-z_0)y^+(z)}\qquad z_0\in\mathcal{L}_g$$

and

$$\mathcal{G}'_+(z_0)-\mathcal{G}'_-(z_0)=0,\quad \lambda\in(0,1)\backslash\cup_{j=1}^{g+1}(u_{2j},u_{2j-1}).$$

Therefore because of the assumption of the continuity of $\mathcal{G}'_\pm$ on $(0,1)$, the following relation must be satisfied

$$\lim_{z_0\to u_i}\mathcal{G}'(z_0)=0,\quad i=1,\dots,2g+1, \tag{4.18}$$

where $z_0\in(u_{i+1},u_i)$ for i odd and $z_0\in(u_{i-1},u_i)$ for i even.
We consider analytic initial data or smooth initial data in $(0,1)$ that satisfies (3.1), so that the function $\rho'(\lambda)$ is Hölder continuous in subsets of $(0,1)$, namely

$$|\rho'(\lambda_1)-\rho'(\lambda_2)|<c|\lambda_1-\lambda_2|^\delta,\quad \forall\lambda_1,\lambda_2\in J$$

where the constant $c>0$, $0<\delta\le 1$, and J is some open subset of $(0,1)$.

Lemma 4.1 *[22] If the function $\rho'(\lambda)$ is Hölder continuous near and at u_i, $i=1,\dots,2g+1$, then*

$$\lim_{z_0\to u_i}\mathcal{G}'(z_0)=0,\quad i=1,\dots,2g+1,$$

where $z_0\in(u_{i+1},u_i)$ for i odd and $z_0\in(u_{i-1},u_i)$ for i even.

The above lemma guarantees the consistency of the assumptions (4.11) and (4.18).
These considerations suggests to build a second solution $\tilde{\mathcal{G}}'(\lambda)$ of the RH problem (4.10) requiring that $\tilde{\mathcal{G}}'(\lambda)$ satisfies (4.12) and (4.14) while we impose the continuity (4.11) as constraint. The solution $\tilde{\mathcal{G}}'(\lambda)$ is given by the expression

$$\tilde{\mathcal{G}}'(\lambda) = \frac{1}{(2\pi i)y(\lambda)}\int_{\mathcal{L}_g}\frac{2y^+(z)(\rho'(z)-2\alpha'(z))dz}{z-\lambda} - \frac{Q_{g-1}(\lambda)}{y(\lambda)}. \tag{4.19}$$

The polynomial $Q_{g-1}(\lambda)$ has degree $g-1$ and its coefficients are uniquely determined from (4.14) or (4.13). It is easy to verify that

$$\frac{Q_{g-1}(\lambda)}{y(\lambda)} = \frac{1}{2\pi i}\sum_{k=1}^{g}\frac{\phi_k(\lambda)}{d\lambda}\int_{\mathcal{L}_g}dz(2\rho'(z)-2\alpha'(z))\int_{\alpha_k}\frac{y^+(z)d\eta}{y(\eta)(z-\eta)}, \tag{4.20}$$

where $\phi_k(\lambda)$, $k=1,\dots g$ is the basis of holomorphic differential defined in (2.3).

Lemma 4.2 *The function $\tilde{\mathcal{G}}'(\lambda)$ satisfies the conditions (4.12) and (4.14).*

The continuity on the function $\tilde{\mathcal{G}}'(\lambda)$ is obtained imposing that the end points $u_1,\dots,u_{2g+1}$ evolve according to the equations

$$\tilde{\mathcal{G}}'(\lambda)_{\lambda=u_i} = 0, \quad i=1,\dots,2g+1. \tag{4.21}$$

The next theorem establishes the equivalence between the two different solutions $\mathcal{G}'(\lambda)$ and $\tilde{\mathcal{G}}'(\lambda)$ of the RH (4.10).

Theorem 4.3 *The function function $\tilde{\mathcal{G}}'(\lambda)$ and the set of algebraic equations (4.21) is equivalent to the function $\mathcal{G}'(\lambda)$ and the set of algebraic equations (4.17) and (4.13). The equivalence is established for any C^∞ initial data satisfying (3.1).*

Proof: We write $\tilde{\mathcal{G}}'(\lambda)$ in the form

$$\tilde{\mathcal{G}}'(\lambda) = \frac{1}{y(\lambda)}\int_{\mathcal{L}_g}\frac{y^2(z)(2\rho'(z)-2\alpha'(z))dz}{(2\pi i)y^+(z)(z-\lambda)} - \sum_{l=1}^{g}\frac{\phi_l(\lambda)}{d\lambda}\int_{\mathcal{L}_g}dz\frac{(2\rho'(z)-2\alpha'(z))}{(2\pi i)y^+(z)}\int_{\alpha_l}\frac{y^2(z)d\eta}{y(\eta)(z-\eta)}. \tag{4.22}$$

Using the identity

$$\frac{z^k}{z-\lambda} = \sum_{j=0}^{k-1}z^j\lambda^{k-1-j} + \frac{\lambda^k}{z-\lambda}, \tag{4.23}$$

we obtain

$$\frac{y^2(z)}{z-\lambda} = \frac{y^2(\lambda)}{z-\lambda} + \sum_{k=0}^{2g}(-)^k s_{2g-k}\sum_{j=0}^{k}z^j\lambda^{k-j} \tag{4.24}$$

where the s_k's are the symmetric function in the variables $u_1, u_2, \dots u_{2g+1}$, namely $s_0 = 1$, $s_1 = \sum_{k=1}^{2g+1} u_k$, $s_2 = \sum_{k<j} u_k u_j$ and so on. Using (4.24) we can rewrite (4.22) in the form

$$\begin{aligned}
\tilde{\mathcal{G}}'(\lambda) =& \frac{y(\lambda)}{2\pi i}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))dz}{y^+(z)(z-\lambda)} - \sum_{l=1}^{g}\frac{\phi_l(\lambda)}{(2\pi i)d\lambda}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))}{y^+(z)}dz\int_{\alpha_l}\frac{y(\eta)}{z-\eta}d\eta \\
&+\frac{1}{(2\pi i)y(\lambda)}\sum_{k=0}^{2g}(-)^k s_{2g-k}\sum_{j=0}^{k}\lambda^{k-j}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))z^j}{y^+(z)}dz \\
&-\frac{1}{2\pi i}\sum_{l=1}^{g}\frac{\phi_l(\lambda)}{d\lambda}\sum_{k=0}^{2g}(-)^k s_{2g-k}\sum_{j=0}^{k}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))z^j}{y^+(z)}dz\int_{\alpha_l}\frac{\eta^{k-j}}{y(\eta)}d\eta.
\end{aligned} \tag{4.25}$$

Imposing (4.21), we can see that the first term in (4.25) is automatically zero at the branch points by lemma 4.1. Therefore, using (4.25), the equations (4.21) imply

$$\begin{aligned}
&\Bigg[-\sum_{l=1}^{n}\sum_{m=0}^{n-1}\gamma_l^m\lambda^m\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))}{y^+(z)}dz\int_{\alpha_l}\frac{y(\eta)}{z-\eta}d\eta \\
&+\sum_{k=0}^{2g}(-)^k s_{2g-k}\sum_{j=0}^{k}\lambda^{k-j}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))z^j}{y^+(z)}dz \\
&-\sum_{l=1}^{g}\sum_{m=0}^{g-1}\gamma_l^m\lambda^m\sum_{k=0}^{2g}(-)^k s_{2g-k}\sum_{j=0}^{k}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))z^j}{y^+(z)}dz\int_{\alpha_l}\frac{\eta^{k-j}}{y(\eta)}d\eta\Bigg]\Bigg|_{\lambda=u_i} = 0,
\end{aligned}$$

for $i = 1, \dots, 2g+1$. The above quantity is a polynomial in the λ variable of degree $2g$ that must have $2g+1$ zeros, therefore it is identically zero. From the coefficients of degree g to $2g$ we get the moments conditions (4.17). For the coefficients of degree m, $m = 0, \dots, g-1$ we get the relations

$$\begin{aligned}
&-\sum_{l=1}^{g}\gamma_l^m\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))}{y^+(z)}dz\int_{\alpha_l}\frac{y(\eta)}{z-\eta}d\eta + \sum_{k=m+g+1}^{2g}(-)^k s_{2g-k}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))z^{k-m}}{y^+(z)}dz \\
&-\sum_{l=1}^{g}\gamma_l^m\sum_{k=g+1}^{2g}(-)^k s_{2g-k}\sum_{j=g+1}^{k}\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))z^j}{y^+(z)}dz\int_{\alpha_l}\frac{\eta^{k-j}}{y(\eta)}d\eta \equiv 0, \quad m = 0, \dots, g-1.
\end{aligned}$$

Using (2.4) the above relation simplifies to the form

$$\sum_{l=1}^{g}\gamma_l^m\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))}{y^+(z)}dz\int_{\alpha_l}\frac{y(\eta)}{z-\eta}d\eta \equiv 0, \quad m = 0, \dots, g-1. \tag{4.26}$$

Because the matrix $\{\gamma_l^m\}$, $l = 1, \dots, g$, $m = 0, \dots, g-1$ is invertible, the relation (4.26) is equivalent to

$$\int_{\mathcal{L}_g}\frac{(2\rho'(z)-2\alpha'(z))}{y^+(z)}dz\int_{I_l}\frac{y(\eta)}{z-\eta}d\eta \equiv 0, \quad l = 1, \dots, g \tag{4.27}$$

which coincides with the normalization conditions (4.13). Therefore on the solution of (4.17) and (4.13) the function $\tilde{\mathcal{G}}'(\lambda)$ reads

$$\tilde{\mathcal{G}}'(\lambda) = \frac{y(\lambda)}{2\pi i}\int_{\mathcal{L}_g}\frac{(2\rho'(z) - 2\alpha'(z))dz}{y^+(z)(z-\lambda)} = \mathcal{G}'(\lambda). \tag{4.28}$$

□

In the same way we can prove that $\mathcal{G}'(\lambda)$, the moment conditions (4.17) and the normalization conditions (4.13) are equivalent to $\tilde{\mathcal{G}}'(\lambda)$ and the equations (4.21). We remark that the equivalence between the two solutions does not depend on the fact that the problem has a priori a unique solution as follows from the Lax-Levermore theory but it derives only on the structure of the RH problem for the scalar function $\mathcal{G}$.

We observe that using $\omega_z^g(\lambda)$, the meromorphic analogue of the Cauchy kernel defined in (2.10), we can write $\tilde{\mathcal{G}}'(\lambda)d\lambda$ in the form

$$\tilde{\mathcal{G}}'(\lambda)d\lambda = -xd\tilde{p}^g(\lambda) - td\tilde{q}^g(\lambda) + \Omega^g(\lambda), \tag{4.29}$$

where

$$\begin{aligned} d\tilde{p}^g(\lambda) &= -\frac{1}{2\pi i}\int_{\mathcal{L}_g}\omega_z^g(\lambda)z^{-\frac{1}{2}} \\ d\tilde{q}^g(\lambda) &= -\frac{12}{2\pi i}\int_{\mathcal{L}_g}\omega_z^g(\lambda)z^{\frac{1}{2}} \end{aligned} \tag{4.30}$$

and

$$\Omega^g(\lambda) = -\frac{1}{2\pi i}\int_{\mathcal{L}_g}2\omega_z^g(\lambda)\rho'(z). \tag{4.31}$$

Here and below the integrals in the z variable are taken on the upper side of $\mathcal{L}_g$.

Using the above representation we compute the constants Ω_k defined in (3.7)

$$\Omega_k = \int_{\beta_k}\tilde{\mathcal{G}}'(\lambda)d\lambda = -\int_{\beta_k}\int_{\mathcal{L}_g}\omega_z^g(\lambda)\frac{(2\rho'(z) - 2\alpha'(z))}{2\pi i}dz \quad k = 1,\dots,g. \tag{4.32}$$

Integrating by parts the above identity and using (2.15) and (2.14) we obtain

$$\Omega_k = 4\int_{\mathcal{L}_g}\phi_k(z)((\rho(z) - \alpha(z))dz, \quad k = 1,\dots,g.$$

From the above it follows that $\Omega_{g+1} = 0$.

The following theorem due to Krichever [23] connects the solution of the set of algebraic equations (4.17) and (4.13) or (4.21), to a solution of the Whitham equations.

Theorem 4.4 *[23] Let us suppose that the u_i's depend on x and t in such a way that the conditions (4.21) are fulfilled. Then $u_i = u_i(x,t)$, $i = 1,\dots,2g+1$, satisfies the Whitham equations*

$$\frac{\partial}{\partial t}u_i = v_i(\vec{u})\frac{\partial}{\partial x}u_i, \quad i = 1,\dots 2g+1, \tag{4.33}$$

where

$$v_i(\vec{u}) = \frac{d\tilde{q}^g(\lambda)}{d\tilde{p}^g(\lambda)}\Big|_{\lambda=u_i}, \quad i = 1, \ldots, 2g+1 \tag{4.34}$$

and the differentials $d\tilde{p}^g(\lambda)$ and $d\tilde{q}^g(\lambda)$ have been defined in (4.30).

We remark that from proposition (2.1) the following identity is easily verified

$$v_i(\vec{u}) = \frac{d\tilde{q}^g(\lambda)}{d\tilde{p}^g(\lambda)}\Big|_{\lambda=u_i} = \frac{dq^g(\lambda)}{dp^g(\lambda)}\Big|_{\lambda=u_i} \quad i = 1, \ldots, 2g+1, \tag{4.35}$$

where $dp^g(\lambda)$ and $dq^g(\lambda)$ have been defined in (2.7). The second expression of the $v_i(\vec{u})$'s in (4.35) is the classical formula for the speeds of the Whitham equations obtained in [3].
We can write the algebraic equations (4.21) in the form of the so called hodograph transformation introduced by Tsarev [7]:

$$x = -v_i(\vec{u})t + \tilde{w}_i(\vec{u}), \quad i = 1, \ldots, 2g+1, \tag{4.36}$$

where

$$\tilde{w}_i(\vec{u}) = \frac{\Omega^g(\lambda)}{d\tilde{p}^g(\lambda)}\Big|_{\lambda=u_i}, \quad i = 1, \ldots, 2g+1 \tag{4.37}$$

and $\Omega^g(\lambda)$ has been defined in (4.31). As a consequence of theorem 4.3, the set of algebraic equations (4.17) and (4.13) is equivalent to the hodograph transformation (4.36).
In the next section we will show that the $\tilde{w}_i(\vec{u})$'s defined in (4.37) coincide with the classical formulas provided in [9] or [10].

5 Solution of the Tsarev system and linear overdetermined system of Euler-Poisson-Darboux type

We first define the Cauchy problem for the Whitham equations. The initial value problem consists of the following. We consider the evolution on the $x-u$ plane of the initial curve $u(x, t=0) = u_0(x)$ according to the zero-phase equation (1.3). The solution $u(x,t)$ of (1.3), with the initial data $u_0(x)$, is given by the characteristic equation

$$x = -6tu + f(u) \tag{5.1}$$

where $f(u)|_{t=0}$ is the inverse function of the initial data $u_0(x)$. The solution $u(x,t)$ in (5.1) is globally well defined only for $0 \le t < t_0$, where $t_0 = \frac{1}{6}\min_{u\in\mathbb{R}}[f'(u)]$ is the time of gradient catastrophe of (5.1). Near the point of gradient catastrophe and for a short time $t > t_0$, the evolving curve is given by a multivalued function with three branches $u_1(x,t) > u_2(x,t) > u_3(x,t)$, which evolve according to the one-phase Whitham equations.

Outside the multivalued region the solution is given by the zero-phase solution $u(x,t)$ defined in (5.1). On the phase transition boundary the zero-phase solution and the one-phase solution are attached C^1-smoothly.
Since the Whitham equations are hyperbolic [24], other points of gradient catastrophe can appear in the branches $u_1(x,t) > u_2(x,t) > u_3(x,t)$ themselves or in $u(x,t)$.

In general, for $t > t_0$, the evolving curve is given by a multivalued function with and odd number of branches $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$, $g \geq 0$. These branches evolve according to the g-phase Whitham equations. The g-phase solutions for *different* g must be glued together in order to produce a C^1-smooth curve in the (x,u) plane evolving smoothly with t. The initial value problem of the Whitham equations is to determine, for almost all $t > 0$ and x, the phase $g(x,t) \geq 0$ and the corresponding branches $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ from the initial data $x = f(u)|_{t=0}$.
The solution of the Whitham equations for a given g is obtained by the so called hodograph transformation introduced by Tsarev [7].

Theorem 5.1 *If $w_i(\vec{u})$, $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$, solves the linear over-determined system*

$$\frac{\partial w_i}{\partial u_j} = \frac{1}{v_i - v_j}\frac{\partial v_i}{\partial u_j}[w_i - w_j], \quad i,j = 1,2,\ldots,2g+1, \;\; i \neq j, \tag{5.2}$$

then the solution $(u_1(x,t), u_2(x,t), \ldots, u_{2g+1}(x,t))$ of the hodograph transformation

$$x = -v_i(\vec{u})\,t + w_i(\vec{u}) \quad i = 1, \ldots, 2g+1\,, \tag{5.3}$$

satisfies system (1.2). Conversely, any solution $(u_1, u_2, \ldots, u_{2g+1})$ of (1.2) can be obtained in this way in a neighborhood (x_0, t_0) where the u_{ix}'s are not vanishing..

To guarantee that the $g-$phase solutions for different g are attached continuously, the following natural boundary conditions must be imposed on $w_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$, $g > 0$.
When $u_l = u_{l+1}$, $1 \leq l \leq 2g$,

$$w_l^g(u_1, \ldots, u_l, u_l, \ldots, u_{2g+1}) = w_{l+1}^g(u_1, \ldots, u_l, u_l, \ldots, u_{2g+1}) \tag{5.4}$$

and for $1 \leq i \leq 2g+1$, $i \neq l, l+1$

$$w_i^g(u_1, \ldots, u_l, u_l, \ldots, u_{2g+1}) = w_i^{g-1}(u_1, \ldots, \hat{u}_l, \hat{u}_l, \ldots, u_{2g+1}). \tag{5.5}$$

The superscript g and $g-1$ in the w_i's specify the corresponding genus and the hat denotes the variable that have been dropped. When $g = 1$ and $u_2 = u_3$ we have that

$$\begin{aligned} w_1(u_1, u_3, u_3) &= f(u_1) \\ w_2(u_1, u_3, u_3) &= w_3(u_1, u_3, u_3)\,, \end{aligned} \tag{5.6}$$

where $f(u)$ is the initial data. Similar conditions hold true when $u_1 = u_2$, namely

$$\begin{aligned} w_3(u_1, u_1, u_3) &= f(u_3) \\ w_1(u_1, u_1, u_3) &= w_2(u_1, u_1, u_3). \end{aligned} \tag{5.7}$$

We remark that the $v_i(\vec{u})$'s satisfy the boundary conditions (5.4-5.5) and for $g=1$ we have

$$v_1(u_1,u_3,u_3) = -6u_1, \quad v_3(u_1,u_1,u_3) = -6u_3.$$

The solution of the boundary value problem (5.2), (5.4-5.7) has been obtained in [17] for any smooth monotonically increasing initial data.

Theorem 5.2 *[10] Let be $f(u)$ the inverse function of the smooth initial data $u_0(x,0)$. If the function $q_k = q_k(u_1,u_2,\dots,u_{2g+1})$, $1 \le k \le g$, is the symmetric solution of the linear over-determined system*

$$\begin{cases} 2(u_i-u_j)\dfrac{\partial^2 q_k(\vec{u})}{\partial u_i \partial u_j} = \dfrac{\partial q_k(\vec{u})}{\partial u_i} - \dfrac{\partial q_k(\vec{u})}{\partial u_j}, \quad i\neq j,\ i,j=1,\dots,2g+1, \quad g>0 \\ \\ q_k(\underbrace{u,u,\dots,u}_{2g+1}) = F_k(u) \\ \\ F_k(u) = \dfrac{2^{(g-1)}}{(2g-1)!!} u^{-k+\frac{1}{2}} \dfrac{d^{g-k}}{du^{g-k}}\left(u^{g-\frac{1}{2}} f^{(k-1)}(u)\right), \end{cases} \tag{5.8}$$

with the ordering $1 > u_1 > u_2 > \dots > u_{2g+1} > 0$, then $w_i(\vec{u})$, $i=1,\dots,2g+1$, defined by

$$w_i(\vec{u}) = \frac{1}{P_0^g(u_i)}\left[2\partial_{u_i} q_g(\vec{u}) \prod_{n=1,n\neq i}^{2g+1}(u_i-u_n) + \sum_{k=1}^{g} q_k(\vec{u}) \sum_{n=1}^{k}(2n-1)\tilde{\Gamma}_{k-n} P_{n-1}^g(u_i)\right], \tag{5.9}$$

solves the boundary value problem (5.2), (5.4-5.7). Conversely every solution of (5.2), (5.4-5.7) can be obtained in this way.

In (5.9) the polynomials P_n^g's have been defined in (2.8) and the $\tilde{\Gamma}_k$'s are the coefficient of the expansion for $\xi \to \infty$ of

$$y(\xi) = \xi^{g+\frac{1}{2}}(\tilde{\Gamma}_0 + \frac{\tilde{\Gamma}_1}{\xi} + \frac{\tilde{\Gamma}_2}{\xi^2} + \dots + \frac{\tilde{\Gamma}_l}{\xi^l} + \dots). \tag{5.10}$$

The existence and uniqueness of the solution of the boundary value problem (5.8) has been proved in [11].

Theorem 5.3 *[11] The solution of the boundary value problem (5.8) is unique, symmetric with respect to the variables $u_1,u_2,\dots,u_{2g+1}$ and reads*

$$\begin{aligned} q_k(\vec{u}) =& \frac{1}{C}\int_{-1}^{1}\int_{-1}^{1}\dots\int_{-1}^{1} dz_1 dz_2 \dots dz_{2g}(1+z_{2g})^{g-1}(1+z_{2g-1})^{g-\frac{3}{2}}\dots(1+z_3)^{\frac{1}{2}}(1+z_1)^{-\frac{1}{2}} \times \\ & \frac{F_k(\frac{1+z_{2g}}{2}(\dots(\frac{1+z_2}{2}(\frac{1+z_1}{2}u_1 + \frac{1-z_1}{2}u_2) + \frac{1-z_2}{2}u_3) + \dots) + \frac{1-z_{2g}}{2}u_{2g})}{\sqrt{(1-z_1)(1-z_2)\dots(1-z_{2g})}}, \end{aligned} \tag{5.11}$$

where $C = \prod_{j=1}^{2g} C_j$ *and*

$$C_j = \int_{-1}^{1} \frac{(1+\mu)^{\frac{j}{2}-1}}{\sqrt{1-\mu}} d\mu\,. \tag{5.12}$$

The functions $F_k(u)$ and the solutions $q_k(\vec{u})$, $k = 1,\dots,g$, of the boundary value problem (5.8) satisfy the following relations:

$$\begin{aligned} \partial_u F_k(u) &= \frac{2g+1}{2} F_{k+1}(u) + u\partial_u F_{k+1}(u), \quad k = 1,\dots,g-1, \quad g > 0, \\ \partial_{u_i} q_k(\vec{u}) &= \frac{1}{2} q_{k+1}(\vec{u}) + u_i \partial_{u_i} q_{k+1}(\vec{u}) \quad i = 1\dots,2g+1, \quad k = 1,\dots,g-1 \quad g > 0. \end{aligned} \tag{5.13}$$

Lemma 5.4 *[10] The solution of the Whitham equations described by (5.3) where the* $w_i(\vec{u})$*'s are given by (5.9) is* C^1*-smooth on the phase transition boundaries.*

The next theorem shows the equivalence between the hodograph transformation defined in (4.36) and the one define in (5.3)

Theorem 5.5 *For any smooth monotonically increasing initial data satisfying (3.1), the following identity is satisfied*

$$\tilde{w}_i(\vec{u}) \equiv w_i(\vec{u}), \quad i = 1,\dots 2g+1, \tag{5.14}$$

where the $\tilde{w}_i(\vec{u})$*'s are defined in (4.37) and the* $w_i(\vec{u})$*'s are defined in (5.9).*

Therefore combining theorem 4.3 and and theorem 5.5, we deduce that the hodograph transformation (5.3) is equivalent to the set of algebraic equations (4.17) and (4.13).

The next theorem shows that the hodograph transformation (5.3) can be written in a nice algebraic form. This is the first step to transform the moment conditions (4.17) and the normalization conditions (4.13) into a combination of solutions of linear-overdetermined systems of Euler-Poisson-Darboux type.

Theorem 5.6 *For* $g > 0$ *the hodograph transformation (5.3) where the* $w_i(\vec{u})$*'s are defined in (5.9) is equivalent to the following set of* $2g+1$ *algebraic equations*

$$\begin{aligned} &\sum_{j=1}^{2g+1} \partial_{u_j} q_{g-k}(\vec{u}) - k q_{g-k+1}(\vec{u}) = 0, \quad k = 0,\dots,g-2 \\ &\sum_{j=1}^{2g+1} \partial_{u_j} q_1(\vec{u}) - (g-1) q_2(\vec{u}) - 6\,t = 0, \\ 2&\sum_{j=1}^{2g+1} u_j \partial_{u_j} q_1(\vec{u}) + q_1(\vec{u}) - x - 6\,t \sum_{j=1}^{2g+1} u_j = 0, \end{aligned} \tag{5.15}$$

$$\int_{u_{2k}}^{u_{2k-1}} y(\lambda)\Phi(\lambda;\vec{u})d\lambda = 0, \quad k = 1,\ldots,g \tag{5.16}$$

where the function $\Phi(\lambda;\vec{u})$ *is given by the relation*

$$\Phi^g(\lambda;\vec{u}) = \partial_\lambda \Psi^g(\lambda;\vec{u}) + \sum_{i=1}^{2g+1} \partial_{u_i}\Psi^g(\lambda;\vec{u}). \tag{5.17}$$

The function $\Psi^g(\lambda;\vec{u})$ *satisfies the linear overdetermined system of Euler-Poisson-Darboux type*

$$\begin{cases} \dfrac{\partial}{\partial u_i}\Psi^g(\lambda;\vec{u}) - \dfrac{\partial}{\partial u_j}\Psi^g(\lambda;\vec{u}) = 2(u_i - u_j)\dfrac{\partial^2}{\partial u_i \partial u_j}\Psi^g(\lambda;\vec{u}), \quad i \neq j, \;\; i,j = 1,\ldots 2g+1 \\ \dfrac{\partial}{\partial \lambda}\Psi^g(\lambda;\vec{u}) - 2\dfrac{\partial}{\partial u_j}\Psi^g(\lambda;\vec{u}) = 2(\lambda - u_j)\dfrac{\partial^2}{\partial \lambda \partial u_j}\Psi^g(\lambda;\vec{u}), \quad j = 1,\ldots 2g+1 \\ \Psi^g(\lambda;\underbrace{\lambda,\ldots,\lambda}_{2g+1}) = \dfrac{2^g}{(2g+1)!!}f^{(g)}(\lambda) \end{cases} \tag{5.18}$$

where $f^{(g)}(\lambda)$ *is the gth derivative of the smooth monotonically increasing initial data* $f(u)$*.*

The above boundary value problem can be integrate in the form

$$\begin{aligned} \Psi^g(\lambda;\vec{u}) =& \frac{1}{K}\int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} dz_2 dz_2 \ldots dz_{2g+2}(1+z_{2g+2})^g(1+z_{2g+1})^{g-\frac{1}{2}}\ldots(1+z_3)^{\frac{1}{2}} \times \\ & \frac{f^{(g)}(\frac{1+z_{2g+2}}{2}(\ldots(\frac{1+z_3}{2}(\frac{1+z_2}{2}\lambda + \frac{1-z_2}{2}u_1) + \frac{1-z_3}{2}u_2) + \ldots) + \frac{1-z_{2g+2}}{2}u_{2g+1})}{\sqrt{(1-z_2)(1-z_3)\ldots(1-z_{2g+2})}}, \end{aligned} \tag{5.19}$$

where $K = \prod_{j=2}^{2g+2} C_j$ and the C_j's have been defined in (5.12). The solution obtained is symmetric with respect to the variables $u_1,\ldots,u_{2g+1}$. A similar formula can be obtained for $\Phi^g(\lambda;\vec{u})$.

In the next section we identify the system (5.15) with the moment conditions (4.17) and system (5.16) with the normalization condition (4.13).

Proof of Theorem 5.5.
We show that the $\tilde{w}_i(\vec{u})$'s defined in (4.37) satisfy the Tsarev system (5.2) and the boundary conditions (5.4-5.7). Therfore by theorem 5.2, we obtain $\tilde{w}_i(\vec{u}) = w_i(\vec{u})$. We follow the steps in [9],[23].

The quantities $\tilde{w}_i(u_1,u_2,\ldots,u_{2g+1})$, $i = 1,\ldots 2g+1$, are well defined. Indeed let us write $d\tilde{p}^g(\lambda)$ and $\Omega^g(\lambda)$ in the form $d\tilde{p}^g(\lambda) = \dfrac{\tilde{P}^g(u_i,\vec{u})}{y(\lambda)}d\lambda$ and $\Omega^g(\lambda) = \dfrac{\chi^g(\lambda,\vec{u})}{y(\lambda)}d\lambda$ where

$$\chi^g(\lambda,\vec{u}) = -\frac{1}{\pi i}\left[\int_{\mathcal{L}_g}\frac{y(z)}{\lambda - z}\rho'(z) - \sum_{j=1}^{g}\lambda^{g-j}\int_{\mathcal{L}_g}N_j^g(z)\rho'(z)\right] \tag{5.20}$$

and the N_j^g's have been defined in (2.13). Here and below all the integrals are taken on the upper side of $\mathcal{L}_g$. Then

$$\begin{aligned}\tilde{w}_i(u_1,u_2,\dots,u_{2g+1}) &= \frac{\chi^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})}\\ &= -\frac{1}{\pi i\tilde{P}^g(u_i,\vec{u})}\left[\int_{\mathcal{L}_g}\frac{y(z)}{u_i-z}\rho'(z)-\sum_{j=1}^{g}u_i^{g-j}\int_{\mathcal{L}_g}N_j(z)\rho'(z)\right].\end{aligned}\tag{5.21}$$

Next we show that the $\tilde{w}_i$'s satisfy the Tsarev system (5.2). The differentials $\partial_{u_j}\Omega^g(\lambda)$ and $\partial_{u_j}d\tilde{p}^g(\lambda)$ are normalized Abelian differentials of the second kind with a pole at $\lambda=u_j$ of second order. Because of (4.37) the differential $\dfrac{\partial}{\partial u_j}\Omega^g-\tilde{w}_j\dfrac{\partial}{\partial u_j}d\tilde{p}^g(\lambda)$ is holomorphic and

$$\begin{aligned}0 =&\partial_{u_j}\int_{\alpha_k}(\Omega^g(\lambda)-\tilde{w}_j\,d\tilde{p}^g(\lambda)),\\ 0 =&\int_{\alpha_k}\frac{\partial}{\partial u_j}\Omega^g-\tilde{w}_j\frac{\partial}{\partial u_j}d\tilde{p}^g(\lambda),\quad k=1,\dots,g,\end{aligned}$$

that follows from (2.11). Therefore

$$(\partial_{u_j}\Omega^g(\lambda)-\tilde{w}_j\partial_{u_j}d\tilde{p}^g(\lambda))\equiv 0\tag{5.22}$$

because it is a holomorphic differential having all the α-periods equal to zero.

From (5.22) we obtain

$$\frac{\partial}{\partial u_j}\chi^g(\lambda)-\tilde{w}_j\frac{\partial}{\partial u_j}\tilde{P}^g(\lambda)=-\frac{1}{2}\frac{\chi^g(\lambda)-\tilde{w}_j\tilde{P}^g(\lambda)}{\lambda-u_j},\quad i=1,2,\dots,2g+1.\tag{5.23}$$

We use the above identity to evaluate $\partial_j\tilde{w}_i(\vec{u})$. From (5.21) and (5.23) we obtain

$$\begin{aligned}\frac{\partial}{\partial u_j}\tilde{w}_i(\vec{u}) =&\frac{\partial}{\partial u_j}\frac{\chi^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})},\quad i\neq j\\ =&\frac{\partial_{u_j}\chi^g(u_i,\vec{u})-\tilde{w}_j\partial_{u_j}\tilde{P}^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})}+(\tilde{w}_j-\tilde{w}_i)\frac{\partial_{u_j}\tilde{P}^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})}\\ =&(\tilde{w}_j-\tilde{w}_i)\frac{\partial_{u_j}\tilde{P}^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})}-\frac{1}{2}\frac{\chi^g(u_i,\vec{u})-\tilde{w}_j\tilde{P}^g(u_i,\vec{u})}{(u_i-u_j)\tilde{P}^g(u_i,\vec{u})}\\ =&(\tilde{w}_j-\tilde{w}_i)\frac{\partial_{u_j}\tilde{P}^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})}-\frac{1}{2}\frac{\tilde{w}_i-\tilde{w}_j}{u_i-u_j},\end{aligned}\tag{5.24}$$

which shows that

$$\frac{1}{\tilde{w}_i-\tilde{w}_j}\frac{\partial\tilde{w}_i}{\partial u_j}=-\frac{\partial_{u_j}\tilde{P}^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})}-\frac{1}{2}\frac{1}{u_i-u_j}.\tag{5.25}$$

In particular the above argument also applied to $d\tilde{q}^g(\lambda)$, $d\tilde{p}^g(\lambda)$ and $v_i(\vec{u})$ therefore we also have

$$\frac{1}{v_i - v_j}\frac{\partial v_i}{\partial u_j} = -\frac{\partial_{u_j}\tilde{P}^g(u_i,\vec{u})}{\tilde{P}^g(u_i,\vec{u})} - \frac{1}{2}\frac{1}{u_i - u_j},$$

which when combined with (5.25) proves the Tsarev relation for the $\tilde{w}_i$'s.
Next we show that the functions $\tilde{w}_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, \ldots, 2g+1$, satisfy the boundary conditions (5.4-5.7).

In the following we use the superscript g to denote the corresponding genus of the quantities we are referring to. We need to consider the behavior of the Abelian differential $\omega^g_z(\lambda)$ when two branch points become coincident. For the purpose let be $u_l = v + \sqrt{\delta}$, $u_{l+1} = v - \sqrt{\delta}$ where $0 < \delta \ll 1$. The differential $\omega^g_z(\lambda) = \omega^g_z(\lambda; u_1, \ldots, u_{l-1}, v - \sqrt{\delta}, v + \sqrt{\delta}, \ldots, u_{2g+1})$ has the following expansion for $\delta \to 0$ when l is odd [25]

$$\omega^g_z(\lambda,\delta) = \omega^{g-1}_z(\lambda) + \frac{\delta}{2}\omega^{g-1}_z(v)\partial_v\omega^g_v(\lambda) + O(\delta^2), \tag{5.26}$$

where $\omega^{g-1}_v(\lambda)$ is the normalized Abelian differential of the third kind having first order poles at the points $Q^{\pm}(z) = (z, \pm\tilde{y}(z))$ with residue ± 1 respectively and it is defined on the Riemann surface

$$\mathcal{S}_{g-1} : \tilde{y}^2 = \prod_{\substack{k=1\\ k\neq l,l+1}}^{2g+1}(\lambda - u_k). \tag{5.27}$$

The differential $O(\delta^2)/\delta^2$ has poles at $\lambda = v$ of order at most 4 and zero residue. In formula (5.26) the quantity $\omega^{g-1}_z(v) = \dfrac{\omega^{g-1}_z(\lambda)}{d\lambda}|_{\lambda=v}$.
When l is even $\omega^g_z(\lambda)$ has the following expansion [25] for $\delta \to 0$

$$\omega^g_z(\lambda) \simeq \omega^{g-1}_z(\lambda) - \frac{1}{\log\delta}\,\omega^{g-1}_v(\lambda)\int_{Q^-(v)}^{Q^+(v)}\omega^{g-1}_z(z). \tag{5.28}$$

The above expansion contains also terms of order $\sqrt{\delta}/\log/\delta$. Using (5.26) and (5.28) we can get the expansion of the differentials $\Omega^g(\lambda)$ and $d\tilde{p}^g(\lambda)$ when $u_l = v + \sqrt{\delta}$ and $u_{l+1} = v - \sqrt{\delta}$, $1 \le l \le 2g$. Let be C_v the contour from $v - \sqrt{\delta}$ to $v + \sqrt{\delta}$ on the upper sheet of $\mathcal{S}_g$. When l is odd $\mathcal{L}_g = \mathcal{L}_{g-1} \cup C_v$ where

$$\mathcal{L}_{g-1} = [0, u_{2g+1}] \cup \cdots \cup [u_{l+3}, u_{l+2}] \cup [u_{l-1}, u_{l-2}] \cup \cdots \cup [u_2, u_1]$$

is the corresponding contour defined on the Riemann surface $\mathcal{S}_{g-1}$ of genus $g-1$. From (5.26) we obtain the expansion of $\Omega^g(\lambda) = \Omega^g(\lambda; u_1, \ldots, u_{l-1}, v - \sqrt{\delta}, v + \sqrt{\delta}, \ldots, u_{2g+1})$ as $\delta \to 0$, namely

$$\Omega^g(\lambda) = \Omega^{g-1}(\lambda) + \frac{\delta}{2}\partial_v\omega^{g-1}_v(\lambda)\Omega^{g-1}(v) - \frac{1}{\pi i}\int_{C_v}\omega^g_z(\lambda)\rho'(z)dz + O(\delta^2), \tag{5.29}$$

where

$$\Omega^{g-1}(v) = -\frac{1}{\pi i}\int_{\mathcal{L}_{g-1}} \omega_z^{g-1}(v)\rho'(z)dz.$$

When l is even $\mathcal{L}_g = \mathcal{L}_{g-1}\backslash C_v$ where

$$\mathcal{L}_{g-1} = [0, u_{2g+1}] \cup \cdots \cup [u_{l+4}, u_{l+3}] \cup [u_{l+2}, u_{l-1}] \cup [u_{l-2}, u_{l-3}] \cup \cdots \cup [u_2, u_1]$$

and from (5.28) we obtain the expansion of $\Omega^g(\lambda)$

$$\begin{aligned}\Omega^g(\lambda) &\simeq -\frac{1}{\pi i}\int_{\mathcal{L}_{g-1}} \omega_z^g(\lambda)\rho'(z)dz + \frac{1}{\pi i}\int_{C_v} \omega_z^g(\lambda,\delta)\rho'(z)dz \\ &\simeq \Omega^{g-1}(\lambda) + \frac{1}{\log\delta}\omega_v^{g-1}(\lambda)\int_{Q^-(v)}^{Q^+(v)} \Omega^{g-1}(z) + \frac{1}{\pi i}\int_{C_v} \omega_z^g(\lambda)\rho'(z)dz.\end{aligned} \tag{5.30}$$

The same expansions applies to $d\tilde{p}^g(\lambda)$, therefore combining (5.29) and (5.30) we obtain

$$\left.\frac{\Omega^g(\lambda)}{d\tilde{p}^g(\lambda)}\right|_{\left[\substack{\lambda=u_i \\ u_l=u_{l+1}=v}\right]} = \left.\frac{\Omega^{g-1}(\lambda)}{d\tilde{p}^{g-1}(\lambda)}\right|_{\lambda=u_i}, \quad i \neq l, l+1,\ i = 1, \ldots 2g+1. \tag{5.31}$$

From the above identity it is clear that the boundary conditions (5.5) are satisfied. In order to evaluate (5.31) at the points $\lambda = v \pm \sqrt{\delta}$, we need to do some extra work. Let us defined the quantity

$$\chi^g(\lambda) = y(\lambda)\frac{\Omega^g(\lambda)}{d\lambda}. \tag{5.32}$$

When $u_l = v + \sqrt{\delta}$ and $u_{l+1} = v - \sqrt{\delta}$, l odd, using (5.29) we obtain

$$\begin{aligned}\chi^g(\lambda) =& (\lambda - v)\chi^{g-1}(\lambda) - \frac{y(\lambda)}{\pi i}\int_{C_v} \omega_z^g(\lambda)\rho'(z)dz \\ &+\frac{\delta}{2}\Omega^g(v)(\partial_v\tilde{y}(v) - (\lambda - v)\sum_{k=1}^{g-1}\lambda^{g-1-k}\partial_v N_k^{g-1}(v)) + O(\delta^2),\end{aligned} \tag{5.33}$$

where now $O(\delta^2)/\delta^2$ is a polynomial in λ. Using (5.26) we obtain the following expansion of the second term in the above equation

$$\begin{aligned}\frac{y(\lambda)}{\pi i}\int_{C_v} \omega_z^g(\lambda)\rho'(z)dz =& \frac{1}{\pi i}\int_{C_v} \frac{y^+(z)}{\lambda - z}\rho'(z)dz + \frac{1}{\pi i}\int_{C_v} \frac{z - v}{\sqrt{(z-v)^2 - \delta}}\tilde{y}^+(z)\rho'(z)dz \\ &- \frac{(\lambda - v)}{\pi i}\sum_{k=1}^{g-1}\lambda^{g-1-k}\int_{C_v} N_k^{g-1}(z)\rho'(z)dz + O(\delta)\end{aligned}$$

Therefore

$$\left.\left(\frac{y(\lambda)}{\pi i}\int_{C_v}\omega_z^g(\lambda)\rho'(z)dz\right)\right|_{\lambda=v\pm\sqrt{\delta}} = -\frac{1}{\pi i}\int_{C_v}\tilde{y}^+(z)\rho'(z)\frac{\sqrt{z-v\pm\sqrt{\delta}}}{\sqrt{z-v\mp\sqrt{\delta}}}dz$$
$$-\frac{1}{\pi}\int_{C_v}\frac{z-v}{\sqrt{\delta-(z-v)^2}}\tilde{y}^+(z)\rho'(z)dz$$
$$\pm\frac{\sqrt{\delta}}{\pi i}\sum_{k=1}^{g-1}v^{g-1-k}\int_{C_v}N_k^{g-1}(z)\rho'(z)dz+O(\delta) \tag{5.34}$$
$$=\pm\sqrt{\delta}\,\tilde{y}^+(v)\rho'(v)+\mathcal{R}_1(\delta),$$

where $\lim_{\delta\to 0}\dfrac{\mathcal{R}_1(\delta)}{\sqrt{\delta}}=0$. Combining (5.33) and (5.34) we obtain

$$\chi^g(v\pm\sqrt{\delta})=\pm\sqrt{\delta}(\chi^{g-1}(v)-\tilde{y}^+(v)\rho'(v))+\mathcal{R}_1(\delta). \tag{5.35}$$

In the same way we can get the expansion for $\tilde{P}^g(v\pm\sqrt{\delta})$, namely

$$\tilde{P}^g(v\pm\sqrt{\delta})=\pm\sqrt{\delta}(\tilde{P}^{g-1}(v)-\frac{\tilde{y}^+(v)}{2\sqrt{v}})+O(\delta),$$

which, when combined with (5.35) gives

$$\frac{\chi^g(v\pm\sqrt{\delta})}{\tilde{P}^g(v\pm\sqrt{\delta})}=\frac{\chi^{g-1}(v)-\tilde{y}^+(v)\rho'(v)}{\tilde{P}^{g-1}(v)-\tilde{y}^+(v)/(2\sqrt{v})}+\mathcal{R}_2(\delta), \tag{5.36}$$

where $\lim_{\delta\to 0}\mathcal{R}_2(\delta)=0$.
Using (5.30) we obtain the following expansion of $\chi^g(\lambda)$ when $u_l=v+\sqrt{\delta}$ and $u_{l+1}=v-\sqrt{\delta}$, l even,

$$\chi^g(\lambda)\simeq(\lambda-v)\chi^{g-1}(\lambda)+\frac{y(\lambda)}{\pi i}\int_{C_v}\frac{\omega_z^g(\lambda)}{d\lambda}\rho'(z)dz$$
$$+\frac{1}{\log\delta}(\tilde{y}(v)-(\lambda-v)\sum_{k=1}^{g-1}\lambda^{g-k}N_k^{g-1}(v))\int_{Q^-(v)}^{Q^+(v)}\Omega^{g-1}(z).$$

From the above and (5.34) we can evaluate

$$\chi^g(v\pm\sqrt{\delta})\simeq\pm\sqrt{\delta}(\chi^{g-1}(v)+\tilde{y}(v)\rho'(v))-\frac{1}{\log\delta}(\tilde{y}(v)\pm\sqrt{\delta}\sum_{k=1}^{g-1}v^{g-1-k}N_k^{g-1}(v))\int_{Q^-(v)}^{Q^+(v)}\Omega^{g-1}(z)$$

and

$$\tilde{P}^g(v\pm\sqrt{\delta})\simeq\pm\sqrt{\delta}(\tilde{P}^{g-1}(v)+\frac{1}{2\sqrt{v}}\tilde{y}(v))+\frac{1}{\log\delta}(\tilde{y}(v)\pm\sqrt{\delta}\sum_{k=1}^{g-1}v^{g-1-k}N_k^{g-1}(v))\int_{Q^-(v)}^{Q^+(v)}d\tilde{p}^{g-1}(z).$$

Combining the above two expansions we obtain

$$\frac{\chi^g(v \pm \sqrt{\delta})}{\tilde{P}^g(v \pm \sqrt{\delta})} \simeq \frac{\int_{Q^-(v)}^{Q^+(v)} \Omega^{g-1}(z)}{\int_{Q^-(v)}^{Q^+(v)} d\tilde{p}^{g-1}(z)}. \tag{5.37}$$

The relations (5.36) and (5.37) show that the boundary conditions (5.4) are satisfied. From (5.31) we obtain

$$w_1(u_1, v, v) = \frac{\chi^0(u_1)}{\tilde{P}^0(u_1)} = \frac{\int_0^{u_1} \frac{\rho'(z)}{\sqrt{z-u_1}} dz}{\int_0^{u_1} \frac{1}{2\sqrt{z-u_1}\sqrt{z}} dz} = f(u_1), \tag{5.38}$$

which shows that the boundary condition (5.6) is satisfied. Analogous considerations can be done for proving (5.7). Theorem 5.5 is then proved. □

Proof of Theorem 5.6. Following the steps in [10] we consider the polynomial

$$Z^g(\lambda) := -xP_0^g(\lambda) - 12tP_1^g(\lambda) + R^g(\lambda), \tag{5.39}$$

where $R^0(\lambda) = f(u)$ and $R^g(\lambda)$, $g > 0$, is given by the expression

$$R^g(\lambda) = 2\sum_{k=1}^{2g+1} \partial_{u_k} q_g(\vec{u}) \prod_{l=1, l\neq k}^{2g+1} (\lambda - u_l) + \sum_{k=1}^{g} q_k(\vec{u}) \sum_{l=1}^{k} (2l-1)\tilde{\Gamma}_{k-l} P_{l-1}^g(\lambda), \tag{5.40}$$

with the polynomials $P_l^g(\lambda)$, $l \geq 0$, defined in (2.8) and the functions $q_k(\vec{u})$, $k = 1, \ldots, g$, defined in (5.8). Then the hodograph transformation (5.3) is equivalent, for $g > 0$, to the equation

$$Z^g(\lambda) \equiv 0, \quad g > 0. \tag{5.41}$$

The proof of the above proposition is obtained observing that the $w_i(\vec{u})$'s defined in (5.9) are given by the ratio $w_i(\vec{u}) = \dfrac{R^g(u_i)}{P_0^g(u_i)}$, $i = 1, \ldots, 2g+1$, where $R^g(\lambda)$ is the polynomial defined in (5.40). Hence we can write the hodograph transformation (5.3) in the form

$$[-xP_0^g(\lambda) - 12tP_1^g(\lambda) + R^g(\lambda)]_{\lambda = u_i} = 0, \quad i = 1, \ldots, 2g+1. \tag{5.42}$$

For $g > 0$, $Z^g(\lambda)$ is a polynomial of degree $2g$ and because of (5.42) it must have at least $2g+1$ real zeros. Therefore it is identically zero. Putting equal to zero the first $g+1$ coefficients of the polynomial $Z^g(\lambda)$ and using repeatedly the identity (5.13) we obtain (5.15).
Putting to zero the coefficients of $Z^g(\lambda)$ of degree 0 to degree $g-1$ is equivalent to the equations

$$\int_{\alpha_k} \frac{Z^g(\lambda)}{y(\lambda)} d\lambda = 0, \quad k = 1, \ldots, g \tag{5.43}$$

Indeed when (5.15) is satisfied, the differential $\dfrac{Z^g(\lambda)}{y(\lambda)}d\lambda$ is an holomorphic differential and by (5.43) it has all its alpha-periods equal to zero. Therefore it is identically zero. Hence the last g equations reads

$$\begin{aligned}0 &= \int_{\alpha_k}\frac{Z^g(\lambda)}{y(\lambda)}d\lambda = \int_{\alpha_k}\frac{-xP_0^g(\lambda) - 12tP_1^g(\lambda) + R^g(\lambda)}{y(\lambda)}d\lambda \\ &= \int_{\alpha_k} 2\frac{\sum_{i=1}^{2g+1}\partial_{u_i}q_g(\vec{u})\prod_{j=1,j\neq i}^{2g+1}(\lambda - u_j)}{y(\lambda)}d\lambda, \quad k = 1,\dots,g.\end{aligned} \tag{5.44}$$

In the third equality of (5.44) we have used the fact that

$$\int_{\alpha_k}\frac{P_l^g(\lambda)}{y(\lambda)}d\lambda = \int_{\alpha_k}\sigma_l^g(\lambda) = 0, \quad l \geq 0,\ k = 1,\dots,g,$$

because of the normalization conditions (2.6). The function $\Psi^g(\lambda;\vec{u})$ defined in (5.19) satisfies the relations

$$\frac{\Psi^g(\lambda;\vec{u})}{\lambda - u_i} - \frac{\Psi^g(u_i;\vec{u})}{\lambda - u_i} = 2\partial_{u_i}\Psi^g(\lambda;\vec{u}), \quad 2\partial_{u_i}q_g(\vec{u}) = \Psi^g(u_i;\vec{u}). \tag{5.45}$$

Using (5.45) we can rewrite the last term in (5.44) in the form

$$\begin{aligned}0 =& \int_{\alpha_k} 2\frac{\sum_{i=1}^{2g+1}\partial_{u_i}q_g(\vec{u})\prod_{j=1,j\neq i}^{2g+1}(\lambda - u_j)}{y(\lambda)}d\lambda, \quad k = 1,\dots,g, \\ =& 2\int_{u_{2k}}^{u_{2k-1}} y(\lambda)\left(\sum_{i=1}^{g}\frac{\Psi^g(\lambda;\vec{u})}{\lambda - u_i} - 2\partial_{u_i}\Psi^g(\lambda;\vec{u})\right)d\lambda \\ =& -4\int_{u_{2k}}^{u_{2k-1}} y(\lambda)\left(\partial_\lambda\Psi^g(\lambda;\vec{u}) + \sum_{i=1}^{g}\partial_{u_i}\Psi^g(\lambda;\vec{u})\right)d\lambda, \quad k = 1,\dots,g,\end{aligned}$$

where the last equality has been obtained integrating by parts. Using the definition of $\Phi^g(\lambda;\vec{u})$ in (5.17) we rewrite the above relation in the form

$$0 = -4\int_{u_{2k}}^{u_{2k-1}} y(\lambda)\Phi^g(\lambda;\vec{u})d\lambda, \quad k = 1,\dots,g, \tag{5.46}$$

which is equivalent to (5.16). □

5.1 The function $\mathcal{G}'$ and linear-overdetermined systems of Euler Poisson Darboux type.

In this section we show that the set of algebraic equations described by the moment conditions (4.17) and the normalization conditions (4.13) can be written in terms of solutions of linear overdetermined system of Euler-Poisson-Darboux type introduced in [11].

Theorem 5.7 *For any monotonically increasing analytic initial data satisfying (3.1), the set of algebraic equations defined by the moment conditions (4.17) is equivalent, for $g > 0$, to (5.15); the set of algebraic equations defined by the normalization conditions (4.13) is equivalent, for $g > 0$, to (5.16), namely*

$$\begin{aligned}
&\sum_{j=1}^{2g+1} \partial_{u_j} q_{g-k}(\vec{u}) - k q_{g-k+1}(\vec{u}) = \frac{1}{\pi i} \int_{\mathcal{L}_g} \frac{(\rho'(z) - a'(z)) z^k}{y^+(z)} dz, \quad k = 0, \dots, g-2 \\
&\sum_{j=1}^{2g+1} \partial_{u_j} q_1(\vec{u}) - (g-1) q_2(\vec{u}) - 6\,t = \frac{1}{\pi i} \int_{\mathcal{L}_g} \frac{(\rho'(z) - a'(z)) z^{g-1}}{y^+(z)} dz \\
&2 \sum_{j=1}^{2g+1} u_j \partial_{u_j} q_1(\vec{u}) + q_1(\vec{u}) - x - 6\,t \sum_{j=1}^{2g+1} u_j = \frac{2}{\pi i} \int_{\mathcal{L}_g} \frac{(\rho'(z) - a'(z)) z^g}{y^+(z)} dz \\
&\int_{u_{2k}}^{u_{2k-1}} 2 y^+(\lambda) \Phi(\lambda; \vec{u}) d\lambda = \int_{I_k} (\mathcal{G}'_+(\lambda) - \mathcal{G}'_-(\lambda)) d\lambda, \quad k = 1, \dots, g.
\end{aligned} \tag{5.47}$$

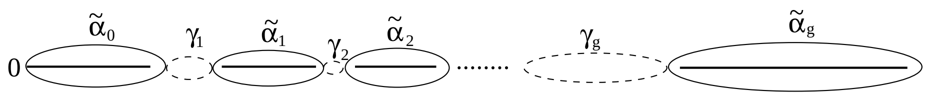
Proof: we first consider the moment conditions. Since $a'(z) = 6tz^{\frac{1}{2}} + \frac{x}{2} z^{-\frac{1}{2}}$ we have

$$\begin{aligned}
&\int_{\mathcal{L}_g} \frac{a'(z) z^k}{y(z)} dz = 0, \quad k = 0, \dots g-2 \\
&\frac{1}{\pi i} \int_{\mathcal{L}_g} \frac{a'(z) z^{g-1}}{y^+(z)} dz = 6t, \quad \frac{2}{\pi i} \int_{\mathcal{L}_g} \frac{a'(z) z^g}{y^+(z)} dz = x + 6t \sum_{j=1}^{2g+1} u_j
\end{aligned} \tag{5.48}$$

For analytic initial data we can deform the integrals

$$\int_{\mathcal{L}_g} \frac{\rho'(z) z^k}{y(z)} dz = \frac{1}{2} \sum_{j=0}^{g} \int_{\tilde{\alpha}_j} \frac{\rho'(z) z^k}{y(z)} dz, \quad k = 0, \dots, g \tag{5.49}$$

where $\tilde{\alpha}_0$ is a close loop around the interval $[0, u_{2g+1})$ passing through zero and the $\tilde{\alpha_j}$ are close loops around the intervals I_j and within the domain of analyticity of the initial data. We can deform the contours $\cup \tilde{\alpha_j}$ to a single contour $\mathcal{C} = \cup_{j=0}^{g} \tilde{\alpha_j} \cup_{j=1}^{g} \gamma_j$ where the contours γ_j are the closed contours plotted in the figure below.



The integrals

$$\int_{\gamma_j} \frac{\rho'(z) z^k}{y(z)} dz = 0, \quad j = 1, \dots, g,$$

therefore

$$\int_{\mathcal{L}_g} \frac{\rho'(z) z^k}{y(z)} dz = \frac{1}{2} \int_{\mathcal{C}} \frac{\rho'(z) z^k}{y(z)} dz, \quad k = 0, \dots, g \tag{5.50}$$

From the right hand side of (5.50) it is straightforward to verify that the integrals $\int_{\mathcal{L}_g} \frac{\rho'(z)z^k}{y(z)}dz$, $k \geq 0$, are symmetric with respect to the variables $u_1, \ldots, u_{2n+1}$ and satisfy the linear overdetermined system of Euler-Poisson-Darboux type defined in (5.8) with the initial data

$$\begin{aligned} \frac{1}{\pi i}\int_{\mathcal{L}_g} \frac{\rho'(z)z^k}{y^+(z)}dz\Big|_{[u_1=u_2=\cdots=u_{2g+1}=u]} &= \partial_u F_{g-k}(u) - kF_{g-k+1}(u), \quad k=0,\ldots,g-1, \\ \frac{2}{\pi i}\int_{\mathcal{L}_g} \frac{\rho'(z)z^g}{y^+(z)}dz\Big|_{[u_1=u_2=\cdots=u_{2g+1}=u]} &= 2u\partial_u F_1(u) + F_1(u), \end{aligned} \tag{5.51}$$

where the functions $F_k(u)$, $k=1,\ldots,g$, have been defined in (5.8). Because of the uniqueness of the solution of the boundary value problem (5.8) the first $g+1$ equations in (5.47) are satisfied.

As regarding the normalization conditions (4.13) we have the following identity for $\lambda \in I_j$, $j = 0,\ldots,g$

$$\begin{aligned} \mathcal{G}'_+(\lambda) - \mathcal{G}'_-(\lambda) =& \frac{2y^+(\lambda)}{\pi i}\left(v.p.\int_{\mathcal{L}_g} \frac{\rho'(z)-a'(z)}{(z-\lambda)y^+(z)}dz\right) \\ &= \frac{y^+(\lambda)}{\pi i}\sum_{j=0}^{g}\int_{\tilde{\alpha_j}} \frac{\rho'(z)-a'(z)}{(z-\lambda)y^+(z)}dz \\ &= \frac{y^+(\lambda)}{\pi i}\int_{\mathcal{C}} \frac{\rho'(z)-a'(z)}{(z-\lambda)y^+(z)}dz \\ &= 2y^+(\lambda)(\Phi^g(\lambda;\vec{u}) - 6t\epsilon_{g0}), \end{aligned} \tag{5.52}$$

where the function $\Phi^g(\lambda;\vec{u})$ has been defined in (5.17) and ϵ_{g0} is equal to one for $g=0$ and zero otherwise. The last identity in (5.52) has been obtained performing computations similar to the ones in (5.51). Therefore

$$0 = \int_{I_j}(\mathcal{G}'_+(\lambda) - \mathcal{G}'_-(\lambda)d\lambda = 2\int_{u_{2j}}^{u_{2j-1}} y^+(\lambda)\Phi^g(\lambda;\vec{u})d\lambda, \quad j=1,\ldots,g,\ g>0 \tag{5.53}$$

and theorem 5.7 in then proved. □

Theorem 5.8 *The relations (5.47) are satisfied for smooth initial data.*

Proof: the proof is obtained combining theorems 4.3, 5.5, 5.6 and 5.7. □

In the following we write the variational conditions in (3.7) and (3.8) in terms of the function $\Phi^g(\lambda;\vec{u})$ defined in (5.17).

Theorem 5.9 *The variational conditions (3.7) and (3.8) can be written in the form*

$$0 > \frac{\mathcal{G}_+(\lambda) - \mathcal{G}_-(\lambda)}{2i} = i\int_{\lambda}^{u_{2j-1}} y^+(\xi)(\Phi^g(\xi;\vec{u}) - 6t\epsilon_{g0})d\xi, \quad \lambda \in (u_{2j}, u_{2j-1}),\ j=1,\ldots,g+1 \tag{5.54}$$

and

$$0 < \mathcal{G}'_+ + \mathcal{G}'_- - 2\rho' + 2\alpha' = 2y(\lambda)(\Phi^g(\lambda;\vec{u}) - 6t\epsilon_{g0}), \quad \lambda \in (u_{2j+1}, u_{2j}),\ j = 0, \dots, g,\ u_0 = 1, \quad (5.55)$$

where the function $\Phi^g(\lambda;\vec{u})$ *has been defined in (5.17) and* ϵ_{g0} *is equal to* 1 *for* $g = 0$ *and zero otherwise.*

Proof: we first prove (5.54). From (5.52) and lemma 4.1 we obtain

$$\frac{\mathcal{G}_+ - \mathcal{G}_-}{2i} = -\int_\lambda^{u_{2j-1}} \frac{\mathcal{G}'_+ - \mathcal{G}'_-}{2i} = i\int_\lambda^{u_{2j-1}} y(\xi)\Phi^g(\xi;\vec{u})d\xi < 0, \quad \lambda \in \cup_{j=1}^{g+1}(u_{2j}, u_{2j-1}),$$

which coincides with (5.54).

As regarding (5.55) we have

$$\mathcal{G}'_+ + \mathcal{G}'_- = \frac{y(\lambda)}{\pi i}\int_{\mathcal{L}_g} \frac{2\rho'(z)}{y^+(z)(z-\lambda)}dz - 2\alpha'(\lambda) - 12t\epsilon_{g0}\, y(\lambda), \quad \lambda \in (u_{2j+1}, u_{2j}),\ j = 0, \dots, g, \quad (5.56)$$

where ϵ_{g0} is equal to one for $g = 0$ and zero otherwise.

In order to express the integral

$$\int_{\mathcal{L}_g} \frac{2\rho'(z)}{y^+(z)(z-\lambda)}dz$$

as the solution of a linear overdetermined system of Euler-Poisson-Darboux type, let us consider the function $q_{g+1}(\tilde{u_1}, \tilde{u_2}, u_1, u_2, \dots, u_{2g+1}) = q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u})$ which satisfies (5.8) with initial data

$$q_{g+1}(\underbrace{u, u, \dots, u}_{2g+3}) = \frac{2^g}{(2g+1)!!} f^{(g)}(u).$$

Then

$$\left(\sum_{j=1}^{2g+1} \partial_{u_j} q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u}) + \partial_{\tilde{u_1}} q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u}) + \partial_{\tilde{u_2}} q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u}) \right)\Bigg|_{\tilde{u_1} = \tilde{u_2} = \lambda} = \Phi^g(\lambda;\vec{u}). \quad (5.57)$$

The derivation of the above identity is straightforward. Next we consider the first identity in (5.47) for the function $q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u})$. Let us suppose $u_{2j} > \tilde{u_1} > \tilde{u_2} > u_{2j+1}$, for $j = 0, \dots, g$, $u_0 = 1$, let us define the interval $\tilde{I} = (\tilde{u_2}, \tilde{u_1})$ and $Y(\lambda) = y(\lambda)\sqrt{(z - \tilde{u_2})(z - \tilde{u_1})}$. The function $Y(\lambda)$ is analytic in the complement of $(-\infty, 0] \cup \mathcal{L}_g \cup I$ and positive for $\lambda > u_1$. $Y^+(\lambda)$ denotes the boundary value from above. Then by (5.47) we have

$$\sum_{j=1}^{2g+1} \partial_{u_j} q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u}) + \partial_{\tilde{u_1}} q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u}) + \partial_{\tilde{u_2}} q_{g+1}(\tilde{u_1}, \tilde{u_2}, \vec{u}) = \frac{1}{\pi i}\int_{\mathcal{L}_g \cup \tilde{I}} \frac{\rho'(z)}{Y^+(z)}dz.$$

The limit $\tilde{u_1} \to \tilde{u_2}$ of the left hand side of the above identity has been obtained in (5.57). Here we derive the limit of the right hand side. Let us define $\tilde{u_1} = \lambda + \sqrt{\delta}$, $\tilde{u_2} = \lambda - \sqrt{\delta}$. Then

$$\lim_{\delta\to 0}\frac{1}{\pi i}\int_{\mathcal{L}_g\cup\tilde{I}}\frac{2\rho'(z)}{Y^+(z)}dz = \frac{1}{\pi i}\int_{\mathcal{L}_g}\frac{2\rho'(z)}{y^+(z)(z-\lambda)}dz + \lim_{\delta\to 0}\frac{1}{\pi i}\int_{\lambda-\sqrt{\delta}}^{\lambda+\sqrt{\delta}}\frac{2\rho'(z)}{Y^+(z)}dz$$
$$=\frac{1}{\pi i}\int_{\mathcal{L}_g}\frac{2\rho'(z)}{y^+(z)(z-\lambda)}dz - \frac{2\rho'(\lambda)}{y(\lambda)}.$$

Combining the previous three relations we obtain

$$2\Phi^g(\lambda;\vec{u}) = \frac{1}{\pi i}\int_{\mathcal{L}_g}\frac{2\rho'(z)}{y^+(z)(z-\lambda)}dz - \frac{2\rho'(\lambda)}{y(\lambda)},\ \ \lambda\in(u_{2j+1},u_{2j}),\ j=0,\dots,g,\ u_0=1. \tag{5.58}$$

Combining (5.56) and (5.58) the variational condition (5.55) can be easily obtained. □

Remark 5.10 *We observe that the function $\Phi^g(\lambda;\vec{u}) - 6t\delta_{g0}$ must be nonzero at the branch points. This is obvious for g=0. Indeed $\Phi^0(u;u) - 6t = f'(u) - 6t = 1/u_x(x,t)$. For $g > 0$ the derivatives $\partial_x u_i(x,t)$ are given by the expression [10]*

$$\partial_x u_i(x,t) = \frac{P_0^g(u_i)}{2\prod_{\substack{j=1\\ j\neq i}}^{2g+1}(u_i-u_j)\Phi^g(u_i;\vec{u})},\quad i=1,\dots,2g+1,\ g>0,$$

where P_0^g has been defined in (2.8). From the above expression it is obvious that the $\partial_x u_i(x,t)$'s are nonsingular if $\Phi^g(u_i;\vec{u}) \neq 0$, $i=1,\dots,2g+1$.

5.1.1 Phase transitions

We derive the equations which determine a change of genus of the solution of the hodograph transformation (5.3) or the set of algebraic equations (4.17) and (4.13). We observe that a transition necessarily occurs when one of the two conditions (5.54) or (5.55) fail to be satisfied. If the conditions (5.54) fail to be satisfied at some point v in the bands (u_{2j}, u_{2j-1}), $j=1,\dots,g+1$, it follows that

$$\begin{aligned}&\int_v^{u_{2j+1}} y(\xi)(\Phi^g(\xi;\vec{u}) - 6t\epsilon_{g0})d\xi = 0,\\ &\Phi^g(v;\vec{u}) - 6t\epsilon_{g0} = 0 \quad v\in(u_{2j},u_{2j-1}),\ j=1,\dots,g+1.\end{aligned} \tag{5.59}$$

Indeed v must be a stationary point of the integral $\int_\lambda^{u_{2j+1}} y(\xi)(\Phi^g(\xi;\vec{u}) - 6t\epsilon_{g0})d\xi$. When we can solve the above system together with the moment conditions (4.17) and the normalization conditions (4.13) for some $t > 0$, we obtain a point $x(t)$, $v(t)$ and $u_1(t) > u_2(t) > \cdots > u_{2g+1}(t)$ of the boundary between the g-phase solution and the $(g+1)-$phase solution.
In the same way if the conditions (5.55) fail to be satisfied at some point v in the gaps (u_{2j}, u_{2j+1}), $j=0,\dots,g$, it follows that

$$\begin{aligned}&\Phi^g(v;\vec{u}) - 6t\epsilon_{g0} = 0\\ &\partial_v\Phi^g(v;\vec{u}) = 0.\end{aligned} \tag{5.60}$$

The above system describes the point of phase transition when a new band is opening.
Systems (5.59) and (5.60) have been obtained in [10] studying directly the hodograph transformation (5.3).

5.2 Lax-Levermore-Venakides functional

Following the steps in [12] we construct the maximizer of the Lax-Levermore-Venakides functional [4],[5]. Let us make the change of variable $\lambda = 1 - \eta^2$, taking the upper complex half plane onto $C\backslash(-\infty, 1]$. The function $\mathcal{G}(\lambda)$ transforms to $\mathcal{F}(\eta) = \mathcal{G}(\lambda)$. The function $\mathcal{F}$ is analytic off the real η axis. We extend the definition of $\mathcal{F}$ onto the lower complex η-plane by the relation $\mathcal{F}(-\eta) = -\mathcal{F}(\eta)$. The RH problem for the function $\mathcal{F}$ is $\mathcal{F}_+ - \mathcal{F}_- = \mathcal{G}_+ + \mathcal{G}_-$ and $\mathcal{F}_+ + \mathcal{F}_- = -(\mathcal{G}_+ - \mathcal{G}_-)sign\eta$. On the real axis we define the function $\psi^*(\eta)$ to be equal to zero outside the interval $[-1, 1]$, while on the interval $(-1, 1)$

$$\psi^*(\eta) = \partial_\eta[\alpha(\lambda(\eta)) - \rho(\lambda(\eta)) + \frac{1}{2}(\mathcal{F}_+ - \mathcal{F}_-)]. \tag{5.61}$$

The function $\psi^*(\eta)$ is the unique maximizer of the functional [5],[12]

$$Q(\psi) = \frac{1}{\pi}[(2a, \psi) + (L\psi, \psi)], \quad \psi \in L^1([0, 1]),\ \psi \leq 0,$$

where

$$L\psi(\eta) = \frac{1}{\pi}\int_0^1 \ln\left|\frac{\eta - \mu}{\eta + \mu}\right| \psi(\mu)d\mu$$

$$a(\eta, x, t) = 4t\eta^3 - x\eta - 6t\eta + \frac{1}{2}\int_{1-\eta^2}^1 \frac{f(\xi)}{\sqrt{\xi + \eta^2 - 1}}.$$

The maximization problem is attacked analytically by solving the variational conditions

$$\begin{aligned} &L\psi(\eta) + a(\eta, x, t) = 0, \quad \text{when } \psi(\eta, x, t) < 0 \\ &L\psi(\eta) + a(\eta, x, t) > 0, \quad \text{when } \psi(\eta, x, t) = 0. \end{aligned} \tag{5.62}$$

Namely if ψ satisfies (5.62), then $\psi = \psi^*$ [4]. In [12] it is shown that the variational conditions (5.62) transform to (3.7) and (3.8). The benefit of the variational formulation is that, due to the convexity of the maximization problem, the uniqueness of ψ^* and hence of $\mathcal{G}$ is guaranteed. Furthermore for each fixed x and t the support of the maximizer in uniquely defined. An important consequence of this result is the following.

Theorem 5.11 *For almost all x and $t \geq 0$ the solution $u_1(x, t) > u_2(x, t) > \cdots > u_{2g+1}(x, t)$ of the hodograph transformation (5.3)*

$$x = -v_i(\vec{u}) + w_i(\vec{u}), \quad i = 1, \ldots, 2g + 1,\ g \geq 0,$$

which satisfies the constraints (3.7) and (3.8) or, equivalently, (5.54) and (5.55), exists for some $g \geq 0$ and it is unique.

As a final result we write the Lax-Levermore-Venakides maximizer as the solution of a linear overdetermined system of Euler-Poisson-Darboux type.

Theorem 5.12 *For $g > 0$ the maximizer $\psi^*(\eta)$ in (5.61) can be written in the form*

$$\psi^*(\eta;\beta_1,\beta_2,\dots,\beta_{2g+1}) = 0, \quad \eta \in (\beta_1,\beta_2)\cup(\beta_3,\beta_3)\cup\dots\cup(\beta_{2g+1},1)$$
$$\psi^*(\eta;\beta_1,\beta_2,\dots,\beta_{2g+1}) = -2\eta\,\Phi^g(1-\eta^2;1-\beta_1^2,1-\beta_2^2,\dots,1-\beta_{2g+1}^2)y(1-\eta^2)$$
$$\text{when } \eta\in[0,\beta_1]\cup_{j=1}^g[\beta_{2j},\beta_{2j+1}],$$

$$\beta_k = \sqrt{1-u_k}, \quad k=1,\dots,2g+1,$$

$$y(1-\eta^2) = \prod_{j=1}^{2g+1}(1-\eta^2-u_j)^{\frac12}$$

and the function Φ^g has been defined in (5.17).

We observe that the condition (5.55) implies $\psi^*(\eta;\beta_1,\beta_2,\dots,\beta_{2g+1}) < 0$ for $\eta\in(0,\beta_1)\cup_{j=1}^g(\beta_{2j},\beta_{2j+1})$ and vice-versa. The proof of the theorem follows directly from (5.55) and (5.61).

Remark 5.13 *All the information on the initial data of the Lax-Levermore maximizer is contained in the the function $\Phi^g(\lambda;\vec{u})$.*
In [10],[17] an upper bound to the genus of the solution of the Whitham equations was provided. We selected initial data such that the maximum number of real zeros of the function $\Phi^g(\lambda;\vec{u}) - 6t\delta_{g0}$ is $2N-g$ for $0\le g\le N$ where N is the supposed upper bound of the genus. Then we showed that, in such a situation, phase transitions from solutions of genus $g\le N$ to solutions of genus $g>N$ do not occur. The proof of the theorem is obtained only studying the Whitham equations and the hodograph transformation (5.3).
In this new representation of the Lax-Levermore-Venakides maximizer, the above result has an easy interpretation. Namely it gives an upper bound to number of intervals where the function $y(\lambda)(\Phi^g(\lambda;\vec{u})-6t\delta_{g0})$ is positive for $0\le g\le N$.

6 Conclusion

In this paper we have studied the Cauchy problem for the KdV equation with small dispersion and with monotonically increasing initial data. We have used the formulation of the Cauchy problem for KdV as a Riemann-Hilbert (RH) problem [18]. The small ϵ-asymptotics is obtained using the steepest descent method for oscillatory Riemann-Hilbert problems introduced in [16]. Analyticity of the initial data is essential for this step.
This procedure leads to a scalar RH problem for a function $\mathcal{G}$ and a set of algebraic equations constrained by algebraic inequalities.

The scalar RH problem for the function $\mathcal{G}$ is well defined even for smooth initial data.
In this paper we have shown that, for smooth monotonically increasing initial data bounded at infinity, the set of algebraic equations obtained through the Deift, Venakides and Zhou approach can be expressed as the solution of a linear overdetermined systems of equations of Euler-Poisson-Darboux type. We have also shown that this set of algebraic equation is equivalent to the set of algebraic equations defined by the hodograph transformation (5.3). The uniqueness of the Lax-Levermore maximization problems guarantees the uniqueness of the solution of the Cauchy problem for the Whitham equations. As a final result we have shown that the Lax-Levermore maximizer can also be expressed as a solution of a linear overdetermined system of Euler-Poisson-Darboux type.
We believe that the above results can be extended to bump-like initial data.

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